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FINITENESS PROPERTIES OF CHEVALLEY GROUPS OVER $\mathbb{F}_q[t]$

BY

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ABSTRACT

Let $\underline{G}$ be a simple Chevalley group of rank n and $\Gamma = \underline{G}(\mathbb{F}_q[t])$. Then the finiteness length of Γ shall be determined by studying the action of Γ on the Bruhat–Tits building X of $\underline{G}\left(\mathbb{F}_q((\frac{1}{t}))\right)$. This is always possible provided that certain subcomplexes of the links of simplices in X are spherical. As a consequence, one obtains that Γ is of type F_{n-1} but not of type FP_n if $\underline{G}$ is of type A_n, B_n, C_n or D_n and $q \geq 2^{2n-1}$.

Introduction

Important finiteness conditions for discrete groups are the almost equivalent (whether they are in fact equivalent, is still not known) properties FP_n and F_n. Recall that a group is said to be of type FP_n iff there exists a projective resolution of the trivial Γ-module $\mathbb{Z}$ starting with $n+1$ finitely generated projective modules. Γ is by definition of type F_n iff there exists an Eilenberg–MacLane complex $K(\Gamma, 1)$ with finite n-skeleton, i.e. (cf. [Br1], ch. VIII, §7) iff Γ is of type FP_n and finitely presentable for $n \geq 2$.

Now some famous results based on reduction theory and due to Raghunathan (cf. [Ra]), respectively to Borel and Serre (cf. [BS1], [BS2]) imply that Γ is always of type F_∞ ($:= F_n$ for all n) if Γ is an arithmetic group or if Γ is an S-arithmetic subgroup of a reductive group which is defined over a number field.

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Things are quite different in the function field case. For example, $\mathrm{SL}_2(\mathbb{F}_q[t])$ is even not finitely generated (cf. [N]). In [Be1], Behr proposed and in [Be2], he gives a proof for a complete classification of all finitely presentable S-arithmetic subgroups of reductive groups over global function fields involving the local ranks of these groups. By the way, the finitely presented S-arithmetic subgroups of arbitrary algebraic groups over number fields were characterized by Abels (cf. [Ab1]).

By contrast, only a few facts are known concerning higher finiteness properties of S-arithmetic groups in the function field case, if the global rank of the reductive group is greater than 0. (Using Harder's reduction theory, Serre settled the anisotropic case in Théorème 4 of [Se]: Here Γ is always "cocompact" and hence of type F_∞ .) In [Stu], Stuhler showed that $\mathrm{SL}_2(\mathcal{O}_S)$ is of type F_{s-1} and not of type FP_s for every S-arithmetic ring $\mathcal{O}_S$ with $\#S = s$. On the other hand, $\mathrm{SL}_{n+1}(\mathbb{F}_q[t])$ is of type F_{n-1} and not of type FP_n, provided that q is "big enough" (cf. [Abr1], [Ab2]; why one is forced by the method of the proof to impose certain restrictions on q, will become clear in section 2.3 below).

In this paper, we will be concerned with the following generalization of the last mentioned result:

CONJECTURE: *If $\underline{G}$ is a simple Chevalley group of rank n , then $\Gamma = \underline{G}(\mathbb{F}_q[t])$ is of type F_{n-1} but not of type FP_n .*

In the following, this conjecture will be proved for the four classical infinite series of Chevalley groups, again provided that q is big compared with n. As in [BS1], [BS2], [Stu], [Abr1], [Ab2] and [Be2], it will be important to study the action of Γ on the space "naturally" associated with Γ, i.e. on the Bruhat–Tits building X of $\underline{G}\left(\mathbb{F}_q((\frac{1}{t}))\right)$ in our case. The quotient X/Γ being non-compact, one has to do some extra work in order to derive finiteness conditions for Γ . The approach of [BS2], namely compactifying X by adjoining the spherical building at infinity, doesn't look very promising in our situation because the stabilizers of the ideal simplices at infinity possess bad finiteness properties. Therefore, the method introduced by Stuhler and also used in [Abr1], [Ab2] and [Be2] will be applied: We filter $X = \bigcup_{d\in\mathbb{N}_0} X_d$ by Γ-invariant subcomplexes X_d with compact quotients X_d/Γ and investigate the homotopy properties of the inclusions $X_d \hookrightarrow X_{d+1}$. This leads to the study of the local structure of X and in particular to the question whether certain subcomplexes of spherical buildings are spherical, themselves. Using the answer to this question given in [AA] and in [Abr2], a criterion of Brown (cf. [Br2]) implies the desired finiteness properties of Γ .

The paper is organized as follows: In Section 1, we put together the facts concerning Bruhat–Tits buildings as far as they are needed here. We start by reviewing some results of [BT1] in sections 1.1 and 1.2. In 1.3 we study the action on Δ_x of the stabilizer $P_{[x[}$ of a ray $[x[\subset X$ with origin x, where Δ_x is the link in X of the simplex generated by x. This is of some relevance, because $P_{[x[}$ acts in the same way on Δ_x as $\Gamma_x = \mathrm{Stab}_\Gamma(x)$, as we shall see in section 1.4. Section 1 may be of some interest in its own right, because it yields a nice example for the interplay between group theoretic, combinatorial and geometric aspects of Bruhat–Tits buildings. Furthermore, a more elementary proof for a theorem of Soulé (cf. [S], Theorem 1) describing the "reduction theory" for the action of Γ on X is indicated in the remarks following Corollary 3.

In Section 2 it is shown how finiteness properties of Γ may be derived from the action of Γ on X . For this purpose, the Γ-invariant filtration $X = \bigcup_{d\in\mathbb{N}_0} X_d$ introduced by Abels in [Ab2] while investigating $\mathrm{SL}_{n+1}(\mathbb{F}_q[t])$ is recalled in section 2.2. In order to study the homotopy properties of $X_d \hookrightarrow X_{d+1}$, it is now essential to determine the "relative links" $\ell k_{X_d}(\rho)$ of certain simplices $\rho \in X_{d+1} \setminus X_d$. In his paper, Abels did this by using some special features of the A_n-case (cf [Ab2], §5). A different approach which works for all Chevalley groups is presented in section 2.3. It uses the action of Γ_ρ on $\ell k_X(\rho)$ and the results of Section 1. It turns out that the subcomplexes $\ell k_{X_d}(\rho)$ of the spherical building $\ell k_X(\rho)$ are among those which are studied in [AA] and [Abr2] (in fact, this is the main reason why these papers were written). In particular, they are homotopy equivalent to bouquets of spheres if $\underline{G}$ is of type A_n, B_n, C_n or D_n and if q is big compared with n. Therefore, by applying a special version of Brown's criterion which is stated in 2.1, we obtain the results announced above in section 2.4.

The main ideas and most parts of the argumentation underlying the present paper were developed during a stay at the University of Bielefeld in 1989/90 when I was a guest of the SFB 343. Now, as the paper is written down at last, it is a pleasure for me to thank this institution for the opportunities it offered to me and Herbert Abels for his kind invitation. Finally, I would like to express my thanks to him and Helmut Behr for several stimulating discussions on the

subject which is treated here.

1. The Bruhat–Tits building

1.1 PRELIMINARIES. Unfortunately, we shall need a lot of notations throughout this paper. To begin with, we denote by

V	an n-dimensional real vector space, endowed with an inner product $(\cdot,\cdot) : V \times V \longrightarrow \mathbb{R}$, identified with its dual space V^* by means of $(\cdot,\cdot)$ and sometimes also considered as an affine space
Φ	an irreducible and reduced root system (of rank n) in V
Π	a base of Φ
Φ^+	the corresponding subset of all positive roots of Φ ; $\Phi^- = -\Phi^+$
${}^vW = W(\Phi)$	the (finite) linear Weyl group of Φ
$W = W_{\mathrm{aff}}(\Phi)$	the (infinite) affine Weyl group of Φ
$\underline{G} = \underline{G}(\Phi)$	a simply connected Chevalley group of type Φ, defined over $\mathbb{Z}$
$\underline{T}$	a maximal torus of $\underline{G}$ with normalizer $\underline{N}$
$\underline{U}_a$	the one-dimensional unipotent subgroup of $\underline{G}$ associated to $a \in \Phi$
$x_a : \underline{\mathrm{Add}} \xrightarrow{\sim} \underline{U}_a$	the corresponding isomorphism ($\underline{\mathrm{Add}}$ = additive group)
$\underline{U}^+ = \prod_{a\in\Phi^+} \underline{U}_a$	and $\underline{U}^- = \prod_{a\in\Phi^-} \underline{U}_a$
$\underline{G} \hookrightarrow \mathrm{SL}_r$	a faithful representation of $\underline{G}$ such that $\underline{T}$ becomes diagonal, $\underline{U}^+$ upper and $\underline{U}^-$ lower triangular
K	a (commutative) field endowed with a discrete valuation $\omega : K \twoheadrightarrow \mathbb{Z} \cup \{\infty\}$

$\mathcal{O} = \{\lambda \in K \mid \omega(\lambda) \geq 0\}$	the corresponding discrete valuation ring
π	a prime element in $\mathcal{O}$, i.e. $\omega(\pi) = 1$
$k = \mathcal{O}/\pi\mathcal{O}$	the residue class field associated to (K, ω)
$G = \underline{G}(K) \subseteq \mathrm{SL}_r(K)$	the group of K-rational points of $\underline{G}$
$\underline{G}(R) = G \cap \mathrm{SL}_r(R)$	the subgroup of G associated to a subring $R \subseteq K$

Further notations are

$T = \underline{T}(K), \quad H = \underline{T}(\mathcal{O}), \quad N = \underline{N}(K) = N_G(T),$

$U_a = \underline{U_a}(K) = \{x_a(\lambda) \mid \lambda \in K\}$ $(a \in \Phi)$ and

$B = \underline{U}^+(\mathcal{O}) H \underline{U}^-(\pi\mathcal{O})$.

$(T, (U_a)_{a\in\Phi})$ constitutes a generating root datum ("donnée radicielle") in G (cf. [BT1], exemple 6.1.3 b)). Furthermore, ω induces a valuation φ of $(T, (U_a)_{a\in\Phi})$ (cf. [BT1], exemple 6.2.3 b)). According to [BT1], section 6.5 (compare also [IM], §2), G possesses a "double Tits system". Note that the group denoted by G' there coincides with G in our situation because $\underline{G}$ is simply connected. In particular, the groups B and N introduced above constitute a BN-pair for G with Weyl group $N/H = W$.

We denote by $\mathbf{\Delta} = \mathbf{\Delta}(\mathbf{G}, \mathbf{B})$ (notation as in [Br3], section V.3) the associated affine building, by C_0 the "standard chamber" fixed by B and by Σ_0 the "standard apartment" stabilized by N . The geometric realization $|\Sigma_0|$ of Σ_0 will be identified with V in such a way that the vertex with stabilizer $\underline{G}(\mathcal{O})$ becomes the origin of V and that the cone $\bigcup_{\lambda\geq 0} \lambda\,|C_0|$ is equal to the closure of the Weyl chamber corresponding to Π .The set $|\Delta|$ will be endowed with the metric introduced in [BT1], section 2.5. The metric space $\mathbf{X} = |\mathbf{\Delta}|$ is called the **Bruhat–Tits building of G** (relative to φ). Note that the topology of X coincides with the usual CW-topology iff Δ is locally finite, i.e. iff k is finite.

1.2 Stabilizers and links. Recall that the set of hyperplanes

$$\mathcal{H} = \{L_{a,m} \mid a \in \Phi,\ m \in \mathbb{Z}\},$$

where $L_{a,m} = \{x \in V \mid (a, x) + m = 0\}$, induces a partition of V into cells ("facettes"). For every cell $F \subseteq V$ we denote by $\overline{F}$ its closure, by $\sigma_F \in \Sigma_0$ the simplex with $|\sigma_F| = \overline{F}$ and by P_F the stabilizer $P_F := \mathrm{Stab}_G(F) = \mathrm{Stab}_G(\overline{F}) = \mathrm{Stab}_G(\sigma_F)$. To every point $x \in V$ we associate the cell F_x containing x, the simplex $\sigma_x = \sigma_{F_x}$ and the stabilizer $\mathbf{P_x} = \mathbf{Stab_G(x)} = \mathbf{P_{F_x}}$. If we define $U_{a,\ell} := \{x_a(\lambda) |\, \omega(\lambda) \geq \ell\}$ $(a \in \Phi,\ \ell \in \mathbb{R})$, we obtain (cf. [BT1], Proposition 6.4.9) $U_a \cap P_x = U_{a,-(a,x)}$ and

$$(1) \qquad P_x = H\langle U_{a,-(a,x)} |\, a \in \Phi\rangle = \prod_{a\in\Phi^+} U_{a,-(a,x)} \cdot \prod_{a\in\Phi^-} U_{a,-(a,x)} \cdot (N \cap P_x).$$

Furthermore, there exists a normal subgroup $P_x^* \trianglelefteq P_x$ such that $U_a \cap P_x^* = \{x_a(\lambda) |\, \omega(\lambda) > -(a,x)\} =: U_{a,-(a,x)+}$ for all $a \in \Phi$ and

$$(2) \qquad P_x^* = (H \cap P_x^*)\langle U_{a,-(a,x)+} |\, a \in \Phi\rangle = \prod_{a\in\Phi^+} U_{a,-(a,x)+} \cdot \prod_{a\in\Phi^-} U_{a,-(a,x)+} \cdot (H \cap P_x^*)$$

(cf. [BT1], 6.4.23, 6.4.27 and 7.2.7). Define $\overline{G}_x := P_x/P_x^*$, denote by $\rho_{\mathbf{x}} : \mathbf{P_x} \twoheadrightarrow \overline{\mathbf{G}}_{\mathbf{x}}$ the associated epimorphism and set

$$\Phi_x := \{a \in \Phi |\, (a,x) \in \mathbb{Z}\}, \quad \overline{T} := \rho_x(H), \quad \overline{U}_a := \rho_x\left(U_{a,-(a,x)}\right)$$

for all $a \in \Phi_x$.

Applying Proposition 6.4.23 of [BT1], we get:

LEMMA 1: *$(\overline{T}, (\overline{U}_a)_{a\in\Phi_x})$ is a generating root datum of type Φ_x in $\overline{G}_x$.*

Lemma 1 implies, as we shall see immediately, that the subgroup $\overline{G}_x^0 := \langle \overline{U}_a |\, a \in \Phi_x\rangle$ of $\overline{G}_x$ is a Chevalley group in the sense of [St]. First of all, we have

$$\overline{U}_a \cong U_{a,-(a,x)}/U_{a,-(a,x)+1} \cong \underline{\mathrm{Add}}(k) \quad \text{for all} \quad a \in \Phi_x .$$

Set $x_a(\overline{\lambda}) := \rho_x(x_a(\pi^{-(a,x)}\lambda))$ for $\lambda \in \mathcal{O}$, $\overline{\lambda} \in k$ and verify the usual Steinberg relations, those corresponding to Steinberg symbols included (cf. [St], §6). In this way we obtain a well-defined epimorphism $f : \underline{G_x}(k) \twoheadrightarrow \overline{G}_x^0$, where $\underline{G_x}$ denotes the simply connected Chevalley group (scheme) of type Φ_x. Because f is compatible with the respective root data of $\underline{G_x}(k)$ and $\overline{G}_x^0$, parabolics which are mapped onto minimal parabolics by f are minimal, themselves. Therefore, $\ker f$ has to be contained in a minimal parabolic subgroup of $\underline{G_x}(k)$ and hence in the center of $\underline{G_x}(k)$. To summarize:

COROLLARY 1: *If $\underline{G_x}$ denotes the simply connected Chevalley group of type Φ_x then $\underline{G_x}(k)$ is a central extension of $\overline{G}_x^0 = \langle \overline{U}_a |\, a \in \Phi_x \rangle$.*

This group theoretical result may be translated into a statement about the local structure of Δ : Recall that the link of $\sigma \in \Delta$ in Δ is defined to be the subcomplex $\ell k_\Delta(\sigma) := \{\tau \in \Delta |\, \tau \cup \sigma \in \Delta \text{ and } \tau \cap \sigma = \emptyset\}$. If σ is contained in C_0 then $B = \mathrm{Stab}_G(C_0)$ and $N \cap P$ constitute a BN-pair for $P = \mathrm{Stab}_G(\sigma)$. It is easily checked that $\ell k_\Delta(\sigma)$ may be identified with the building $\Delta(P, B)$ in this case. If $x \in V = |\Sigma_0|$ is arbitrary, we choose a chamber $C \in \Sigma_0$ such that $x \in |C|$ and obtain $\mathbf{\Delta_x} := \boldsymbol{\ell k_\Delta(\sigma_x)} \cong \mathbf{\Delta(P_x, P_C)}$ because Δ may be described as $\Delta = \Delta(G, P_C)$ as well. Formulae (1) (applied to an element x' of the interior of $|C|$ which is close to x) and (2) above imply $P_x^* \subseteq P_C$. This allows us to identify $\Delta(P_x, P_C)$ with $\Delta(\overline{G}_x, P_C/P_x^*)$ which is the spherical building associated to $(\overline{T}, (\overline{U}_a)_{a \in \Phi_x})$. Finally, Corollary 1 and the discussion preceding it show that $\Delta(\overline{G}_x, P_C/P_x^*)$ is isomorphic to $\Delta(\Phi_x, k) := \Delta(\underline{G_x}, k)$, the building of $\underline{G_x}$ over k in the sense of [T1], §5. Therefore, we obtain as a further consequence of Lemma 1:

COROLLARY 2: *$\Delta_x = \ell k_\Delta(\sigma_x)$ is isomorphic to the spherical building $\Delta(\Phi_x, k)$ associated to $\underline{G_x}(k)$.*

I will conclude this section by remarking that deeper results concerning group schemes attached to points of X may be found in [BT2], section 4.6.

1.3 SOME GEOMETRY CONCERNING $\overline{D}$. Let D be the Weyl chamber corresponding to Π , i.e. $D = \{z \in V |\, (a, z) > 0 \quad \forall\, a \in \Pi\}$, and let $x \neq 0$ be an element of the closure $\overline{D}$. For reasons which will become clear later (cf. Lemma 5), we are interested in the subgroup $P_{[x[} := \bigcap_{\lambda \geq 1} P_{\lambda x}$ of P_x . In particular, the action of $P_{[x[}$ on Δ_x will be studied. Before doing this, we have to introduce some terminology:

Set $\mathcal{H}_x := \{L_{a,-(a,x)} |\, a \in \Phi_x\} = \{L \in \mathcal{H} |\, x \in L\}$. Choose an open ball U with center x such that $U \cap L = \emptyset$ for all $L \in \mathcal{H} \setminus \mathcal{H}_x$. Then $U \cap (x + D)$ is a nonempty open convex subset of V satisfying $U \cap (x + D) \cap L = \emptyset$ for all $L \in \mathcal{H}$. Hence, there is a unique open n-cell $C = C_{x,D}$ containing $U \cap (x + D)$.

Now consider the ray $[x[:= \{\lambda x | \lambda \geq 1\}$. Then $x \in \overline{D}$ and $U \cap (x + \overline{D}) \subseteq \overline{C}$ imply $[x[\cap \overline{C} \neq \{x\}$. Therefore, we may define a point $y \in [x[\setminus\{x\}$ by setting $[x, y] = [x[\cap \overline{C}$. (I use the notations $[x, y] := \{(1-\lambda)x + \lambda y | \; 0 \leq \lambda \leq 1\}$, $[x, y[:= [x, y] \setminus \{y\}$ and so on.) Furthermore, we introduce the simplices $\sigma_{[x,y]} := \sigma_x \cup \sigma_y$ and $\overline{\boldsymbol{\sigma}}_y := \boldsymbol{\sigma}_{[x,y]} \setminus \boldsymbol{\sigma}_x \in \boldsymbol{\Delta}_x$.

Recall (cf. Lemma 1) that $\overline{G} := \overline{G}_x$ possesses a Tits system, that

$$\overline{B} := \overline{T} \prod_{a \in \Phi^+ \cap \Phi_x} \overline{U}_a = \rho_x(P_C)$$

is a minimal parabolic subgroup of $\overline{G}$ and that $\Delta(\overline{G}, \overline{B})$ is isomorphic to $\Delta_x = \ell k_\Delta(\sigma_x)$. What will be proved next is the fact that the parabolic subgroup $\rho_x(P_{[x[})$ is the stabilizer of $\overline{\sigma}_y$ in $\overline{G}$.

LEMMA 2: *Set* $\Phi_x^0 := \{a \in \Phi | \; (a, x) = 0\}$. *Then*

$$\overline{B}\, W(\Phi_x^0)\overline{B} = \rho_x(P_{[x[}) = \rho_x(P_x \cap P_y) = \mathrm{Stab}_{\overline{G}}(\overline{\sigma}_y) =: \overline{P}_y.$$

Proof: First we list some consequences of [BT1], 6.4.9, 6.4.10 and 7.4.4, namely,

$$\begin{aligned} P_{[x[} &= H\langle U_{a,-(a,x)} | \; a \in \Phi^+ \cup \Phi_x^0 \rangle \\ &= \prod_{a \in \Phi^+} U_{a,-(a,x)} \cdot \prod_{a \in \Phi^- \cap \Phi_x^0} U_{a,0} \cdot (N \cap P_{[x[}) , \\ N \cap P_{[x[} &= N \cap H\langle U_{a,0} | \; a \in \Phi_x^0 \rangle \quad \text{and} \\ & (N \cap P_{[x[})/H \cong W(\Phi_x^0) . \end{aligned}$$

These equations imply $\overline{B}\, W(\Phi_x^0)\overline{B} = \rho_x(P_{[x[})$. The inclusions

$$\rho_x(P_{[x[}) \subseteq \rho_x(P_x \cap P_y) \subseteq \overline{P}_y$$

are obvious. So it suffices to show $\mathrm{rk}\, \overline{B}\, W(\Phi_x^0)\overline{B} = \mathrm{rk}\, \overline{P}_y$, where the "rank" of a parabolic subgroup P is by definition the maximal natural number ℓ such that there exists a strictly increasing chain of parabolics of the form $P_0 \subset P_1 \subset \cdots \subset P_\ell = P$. Hence

$$\mathrm{rk}\; \overline{B}\, W(\Phi_x^0)\overline{B} = \mathrm{rk}\; \Phi_x^0 \left(:= \dim_{\mathbb{R}} \left\{ \sum_{a \in \Phi_x^0} \lambda_a a | \; \lambda_a \in \mathbb{R} \right\} \right) \quad \text{and}$$

$$\mathrm{rk}\; \overline{P}_y = \mathrm{rk}\; \overline{G} - \#\overline{\sigma}_y = \mathrm{rk}\; \Phi_x - \#\overline{\sigma}_y .$$

But $\#\overline{\sigma}_y$, the number of vertices of $\overline{\sigma}_y$, is easy to calculate:

$$\begin{aligned}
\dim|\sigma_x| &= \dim \overline{F}_x = \dim \bigcap_{a\in\Phi_x} L_{a,-(a,x)} \\
&= \dim\Big(x + \bigcap_{a\in\Phi_x} L_{a,0}\Big) = n - \mathrm{rk}\ \Phi_x, \\
\dim|\sigma_{[x,y]}| &= \dim \bigcap_{[x,y]\subseteq L\in\mathcal{H}} L = \dim \bigcap_{[0,x]\subseteq L\in\mathcal{H}} L \\
&= \dim \bigcap_{a\in\Phi_x^0} L_{a,0} = n - \mathrm{rk}\ \Phi_x^0.
\end{aligned}$$

Therefore, $\#\overline{\sigma}_y = \#\sigma_{[x,y]} - \#\sigma_x = \mathrm{rk}\ \Phi_x - \mathrm{rk}\ \Phi_x^0$,

$$\mathrm{rk}\ \overline{P}_y = \mathrm{rk}\ \Phi_x^0 \quad\text{and}\quad \overline{P}_y = \overline{B}\,W(\Phi_x^0)\overline{B}\ .$$

∎

In Section 2, we shall also need a result concerning the construction "opposite" to that we have discussed so far. Let $C^- = C_{x,-D}$ be the n-cell containing $U\cap(x-D)$ and set $]x] := \{\lambda x|\ \lambda \le 1\}$. Define y^- by $[y^-,x] =]x]\cap\overline{C^-}$ and $\overline{\sigma}_{y^-}\in\Delta_x$ by $\overline{\sigma}_{y^-} = \sigma_{[y^-,x]}\setminus\sigma_x$. Note that C^- needn't be contained in $\overline{D}$ but y^- certainly is.

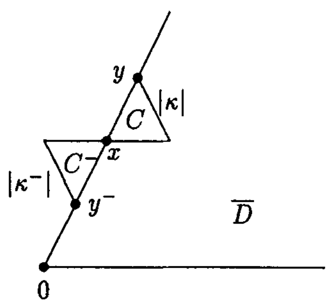

Figure 1

Denote by κ,κ^- the chambers of Δ_x such that $|\kappa|\subseteq\overline{C}$, $|\kappa^-|\subseteq\overline{C^-}$, respectively (see Figure 1, where x is a vertex). Then $\mathrm{Stab}_{\overline{G}}(\kappa)=\overline{B}$, $\mathrm{Stab}_{\overline{G}}(\kappa^-) = \overline{T}\prod_{a\in\Phi^-\cap\Phi_x}\overline{U}_a =: \overline{B}^-$. This shows that κ and κ^- are opposite chambers of Δ_x in the sense of [T1], 3.22. Besides, we obtain:

LEMMA 3: *$\overline{\sigma}_y$ and $\overline{\sigma}_{y^-}$ are opposite in Δ_x .*

Proof: Because

$$\dim|\sigma_{[y^-,x]}| = \dim \bigcap_{[y^-,x]\subseteq L\in\mathcal{H}} L = \dim \bigcap_{[x,y]\subseteq L\in\mathcal{H}} L = \dim|\sigma_{[x,y]}|\ ,$$

the same argumentation as in the proof of Lemma 3 shows

$$\mathrm{Stab}_{\overline{G}}(\overline{\sigma}_{y^-}) = \rho_x(P_{]x]}) = \overline{B}^-\ W(\Phi_x^0)\overline{B}^-\ .$$

Denote by $w_0 \in W(\Phi_x)$ the unique element with $w_0(\Phi^+ \cap \Phi_x) = \Phi^- \cap \Phi_x$. It follows that

$$\mathrm{Stab}_{\overline{G}}(\overline{\sigma}_{y^-}) = w_0(\overline{B}\, w_0^{-1} W(\Phi_x^0) w_0\, \overline{B}) w_0^{-1} \ .$$

Comparing types, we see that this is also the stabilizer of the simplex opposite to $\overline{\sigma}_y$ in the apartment $\Sigma_0 \cap \Delta_x$ (cf. [T1], 2.39, and [R], Lemma 6.1). Hence, $\overline{\sigma}_{y^-}$ is this simplex. ∎

1.4 An alternative description of $\rho_x(\Gamma_x)$. In the following, we consider a more special situation:

$K = k(t)$ is the rational function field over k

$\omega = \omega_\infty$ is defined by $\omega(\frac{f}{g}) := \deg g - \deg f \forall\ 0 \neq g, f \in k[t]$

$\pi = \frac{1}{t}$

$\Gamma = \underline{G}(k[t])$

The notations introduced so far will be kept. Additionally, we set

$$\Gamma_z := \mathrm{Stab}_\Gamma(z) \quad (z \in X \text{ or } z \in \Delta) \quad \text{and} \quad \Gamma_\Omega := \bigcap_{x\in\Omega} \Gamma_x \quad (\Omega \subseteq X)\ .$$

The significance of the sector ("quartier") D in this situation comes from a result which is due to Soulé (cf. [S], Theorem 1):

Lemma 4: *$\overline{D}$ is a fundamental domain for the action of Γ on X and the simplicial complex $F \subset \Sigma_0$ with $|F| = \overline{D}$ is a simplicial fundamental domain for the action of Γ on Δ.*

We shall also need the following

Lemma 5: $\rho_x(\Gamma_x) = \rho_x(P_{[x[})$ *for every* $x \in \overline{D} \setminus \{0\}$.

Proof: By the definition of P_x^* (cf. [BT1], 7.2.7),

$$H = \underline{T}(\mathcal{O}) = \underline{T}(k)\ (H \cap P_x^*)$$

and hence $\rho_x(H) = \rho_x(\underline{T}(k)) \subseteq \rho_x(\Gamma_x) =: \overline{\Gamma}_x$. Furthermore,

$$\overline{U}_a = \rho_x(\{x_a(\pi^{-(a,x)}\lambda) |\ \lambda \in k\}) \subseteq \overline{\Gamma}_x \quad \text{for all } a \in (\Phi^+ \cap \Phi_x) \cup \Phi_x^0.$$

Hence $\rho_x(P_{[x[}) \subseteq \overline{\Gamma}_x$.

We shall show that $\overline{\Gamma}_x$ cannot be strictly bigger than $\rho_x(P_{[x[})$. Suppose it were. Then there would exist a root $b \in (\Phi^- \cap \Phi_x) \setminus \Phi_x^0$ such that $\overline{U}_b \subseteq \overline{\Gamma}_x$. In particular, there were $u^+ \in U^+ \cap P_x^*$, $u^- \in U^- \cap P_x^*$, $h \in H$ and $\gamma \in \Gamma$ such that $x_b(\pi^{-(b,x)})\, u^-\, u^+\, h = \gamma$. An easy calculation using $\mathcal{O}^* \cap k[t] = k^*$ shows

$$U_r^-(K)\, U_r^+(K)\, T_r(\mathcal{O}) \cap \mathrm{SL}_r(k[t]) = U_r^-(k[t])\, U_r^+(k[t])\, T_r(k) .$$

Therefore, we would obtain

$$u := x_b(\pi^{-(b,x)})u^- \in U^- \cap \Gamma = \prod_{a \in \Phi^-} \underline{U_a}(k[t])$$

(the last equation follows from Lemma 49(b) in [St]). On the other hand, $-(b,x) > 0$ and

$$u \in x_b(\pi^{-(b,x)})U_{b,-(b,x)+1} \prod_{a \in \Phi^- \setminus \{b\}} U_a$$

imply $u \notin \prod_{a \in \Phi^-} \underline{U_a}(k[t])$, a contradiction. ∎

The title of this section refers to the following consequence of Lemma 2 and Lemma 5:

COROLLARY 3: $\rho_x(\Gamma_x) = \overline{P}_y$.

Remarks: (i) The origin was excluded in Lemma 5 in order to get a well-defined ray $[x[$. Of course, $\rho_0(\Gamma_0) = \rho_0(P_0) = \rho_0(\underline{G}(\mathcal{O}))$ is easy to prove. Using the definition of $H \cap P_x^*$, it is also possible to identify this group with $\underline{G}(k)$.

(ii) Lemma 5 is an immediate consequence of $\Gamma_x = \Gamma_{[x[}$, an equation derived by Soulé in section 1.1 of [S]. His proof depends on the technically rather complicated §9 of [BT1], where the proof of the result he refers to is only sketched. Therefore, I preferred showing $\rho_x(\Gamma_x) = \rho_x(P_{[x[})$ differently. Having done this, we can easily deduce $\Gamma_x \subseteq P_{[x[}P_x^* \subseteq P_y$, $\Gamma_x \subseteq \Gamma_y$, $\Gamma_x = \Gamma_{[x,y]}$ and, "by induction along $[x[$ ", $\Gamma_x = \Gamma_{[x[}$.

(iii) The identity $\rho_x(\Gamma_x) = \overline{B}\, W(\Phi_x^0)\overline{B}$ is the key argument in the proof of Theorem 1 in [S]. By modifying section 1.3 of [S] as well, it is possible to derive this theorem (which we quoted as Lemma 4) more elementary without referring to §9 of [BT1]. ∎

As was already pointed out by Soulé, a further consequence of $\Gamma_x = \Gamma_{[x[}$ is:

COROLLARY 4:

$$\Gamma_x = \underline{T}(k) \cdot \langle \underline{U}_a(k) | \, a \in \Phi_x^0 \rangle \cdot \prod_{a \in \Phi^+ \setminus \Phi_x^0} \{x_a(f) | \, f \in k[t] \text{and } \deg f \leq (a, x)\}.$$

In particular, Γ_x *is finite iff* k *is finite.*

2. Buildings and finiteness properties of $\underline{G}(\mathbb{F}_q[t])$

2.1 Brown's criterion.

Let Γ be a group and let X be a Γ-CW-complex, i.e. a CW-complex on which Γ acts by homeomorphisms permuting the cells. In this section, we shall recall some conditions derived by K.S. Brown (cf. [Br2], Corollary 3.3(b)) which allow to determine the finiteness length of Γ .

The specialization of Brown's criterion stated below is adapted to the requirements of certain applications. A similar lemma was already discussed in [Abr1], §2 (cf. also [Ab2], Lemma 4.2).

Before stating the criterion, I wish to recall a notion introduced by Quillen (cf. [Q], section 8) which will be crucial in the following:

Definition 1:

(i) An m-dimensional CW-complex is called **m-spherical** iff it is homotopy equivalent to a bouquet of m-spheres, i.e. iff it is either contractible or non-contractible and $(m-1)$-connected.

(ii) A simplicial complex is called m-spherical iff its geometric realization is m-spherical.

LEMMA 6: *Let* X *be a* Γ*-CW-complex. Suppose that the following holds:*

(a) X *is contractible.*

(b) *The stabilizers* Γ_σ *are finite for all cells* σ .

(c) $X = \bigcup_{d \in \mathbb{N}_0} X_d$ *with* Γ*-invariant subcomplexes* X_d *which are finite mod* Γ *for all* d .

(d) $X_{d+1} = X_d \cup \bigcup_{i \in I_d} S_{i,d}$ *with contractible subcomplexes* $S_{i,d}$ *such that*

  (i) $S_{i,d} \cap S_{j,d} \subseteq X_d \quad \forall \, i \neq j, d.$

  (ii) $S_{i,d} \cap X_d$ *is* **(n − 1)–spherical** $\forall \, i, d.$

  (iii) *There exist infinitely many* d *such that* $S_{i,d} \cap X_d$ *is non-contractible for at least one* $i \in I_d$.

Then Γ *is of type* F_{n-1} *but not of type* FP_n .

If $\Gamma = \underline{G}(\mathbb{F}_q[t])$ and if X is the corresponding Bruhat–Tits building then it is well known that condition (a) is satisfied (cf. [BT1], Propsition 2.5.16). The second assumption follows from Corollary 4 (and is of course well known as well). After one has constructed a suitable filtration of X, *condition* (d) (ii) is hardest to verify. This task splits into two parts:

Firstly, one has to describe the intersections $S_{i,d} \cap X_d$ explicitly as subcomplexes of certain spherical buildings. We will do this in section 2.3. Secondly, the homotopy properties of these complexes have to be determined. This part requires completely different methods and will, therefore, be treated elsewhere (cf. [AA] for the A_n-case and [Abr2] for subcomplexes of other spherical buildings).

2.2 Abels' filtration. Let Δ be a building. Fix a chamber C_0 . Denote by $d(C, C')$ the gallery-distance between two chambers C and C' of Δ . Set

$d(\sigma, \sigma') := \min\{d(C, C') |\, \sigma \subseteq C \text{ and } \sigma' \subseteq C'\}$ for simplices $\sigma, \sigma' \in \Delta$ and

$d(A, B) := \min\{d(\sigma, \tau) |\, \sigma \in A \text{ and } \tau \in B\}$ for non-empty subsets $A, B \subseteq \Delta$.
Assume additionally that we are given a group Γ acting (by simplicial automorphisms) on Δ together with a subcomplex $\mathbf{F} \subseteq \mathbf{\Delta}$ containing C_0 which is a **simplicial fundamental domain** with respect to this action. Then there exists a simplicial retraction $r\colon \Delta \longrightarrow F$ mapping every simplex σ onto the unique element of $\Gamma\sigma \cap F$.

Following [Ab2], §2, we define a Γ–invariant filtration of Δ by setting

(3) $\Delta_d := \{\sigma \in \Delta \,|\, d(\sigma, \Gamma C_0) \leq d\}$, $d \in \mathbb{N}_0$.

Note that $d(\sigma, \Gamma C_0) = d(r\sigma, C_0)$ implies

(4) $\Delta_d = \Gamma F_d$ with $F_d := \{\tau \in F |\, d(\tau, C_0) \leq d\}$.

By Lemma 2.4 of [Ab2], every chamber $C \in \Delta_{d+1} \smallsetminus \Delta_d$ contains a simplex $R^\Gamma(C)$, called the "Γ-restriction of C", which satisfies

$$\{\sigma \subseteq C |\, \sigma \in \Delta_d\} = \{\sigma \subseteq C |\, \sigma \not\supseteq R^\Gamma(C)\}.$$

We set $R_{d+1} := \{R^\Gamma(C) |\, C \text{ is a chamber in } \Delta_{d+1} \smallsetminus \Delta_d\}$ and associate to every

$\rho \in R_{d+1}$ the following subcomplexes of Δ :

$$\begin{aligned} S(\rho) &:= st_{\Delta_{d+1}}(\rho) &=& \{\sigma \in \Delta | \, \sigma \cup \rho \in \Delta_{d+1}\}, \\ T'(\rho) &:= \ell k_{\Delta_{d+1}}(\rho) &=& \{\sigma \in S(\rho) | \, \sigma \cap \rho = \emptyset\}, \\ T(\rho) &:= S(\rho) \cap \Delta_d. \end{aligned}$$

LEMMA 7: *With the notations introduced above the following holds:*

(i) $\Delta_{d+1} = \Delta_d \cup \bigcup_{\rho \in R_{d+1}} S(\rho)$,

(ii) $S(\rho) \cap S(\rho') \subseteq \Delta_d \quad \forall \, \rho \neq \rho' \in R_{d+1}$,

(iii) $T(\rho) = \partial\rho * T'(\rho)$, *where* $\partial\rho := \{\sigma \subseteq \rho | \, \sigma \neq \rho\}$ *and "*" means "join",*

(iv) $T'(\rho) \cong T'(r\rho) \quad \forall \, \rho \in R_{d+1}$ *and*
$T'(\rho) = \Gamma_\rho\{\sigma \subseteq \mathrm{proj}_\rho C_0 | \, \sigma \cap \rho = \emptyset\} \quad \forall \, \rho \in R_{d+1} \cap F$.

Proof: Statements (i) – (iii) are due to Abels (cf. [Ab2], Lemma 4.2). The isomorphism in (iv) follows from $T'(\gamma\rho) = \gamma T'(\rho)$ for all $\gamma \in \Gamma$.

Now assume $\rho \in R_{d+1} \cap F$. Denote by $C := \mathrm{proj}_\rho C_0$ the projection of C_0 on ρ (cf. [T1], 3.19), i.e. C is the unique chamber satisfying $C \supseteq \rho$ and $d(C, C_0) = d(\rho, C_0)$. Then $d(\rho, C_0) = d(\rho, \Gamma C_0) = d + 1$ implies $C \in \Delta_{d+1}$ and hence $\Gamma_\rho\{\sigma \subseteq C | \, \sigma \cap \rho = \emptyset\} \subseteq T'(\rho)$.

Conversely, let $\tau \in T'(\rho)$ be given. Choose a chamber $C' \in \Delta_{d+1}$ such that $\tau \cup \rho \subseteq C'$. Then $d(rC', C_0) = d(C', \Gamma C_0) \leq d + 1$ and $\rho \subseteq rC'$ imply $rC' = \mathrm{proj}_\rho C_0 = C$. Hence there exists a $\gamma \in \Gamma$ such that $\gamma C' = C$. Furthermore, $\gamma\rho \subseteq \gamma C' = rC' \in F$ implies $\gamma \in \Gamma_\rho$. This shows

$$\tau = \gamma^{-1}(\gamma\tau) \in \Gamma_\rho\{\sigma \subseteq C | \, \sigma \cap \rho = \emptyset\} . \quad \blacksquare$$

2.3 THE RELATIVE LINKS. We now return to the situation described in section 1.4, assuming additionally $k = \mathbb{F}_q$. In particular, $\Delta = \Delta(G, B)$ is an affine building, $\Gamma = \underline{G}(\mathbb{F}_q[t])$ and F is the simplicial fundamental domain introduced in Lemma 4.

We wish to apply Lemma 6 to the Γ-CW-complex $X = |\Delta|$ by setting $\mathbf{X_d} := |\mathbf{\Delta_d}|$, where Δ_d is defined as in (3). Note that condition (c) follows immediately from (4). Furthermore, Lemma 7 (ii) implies (d) (i) if we define the $S_{i,d}$ to be the $|S(\rho)|$ for $\rho \in R_{d+1}$.

As already mentioned, our next task consists in determining $|S(\rho)| \cap X_d = |T(\rho)|$. In view of Lemma 7 (iii), we only need to know $T'(\rho)$ and in view of 7 (iv), we may assume $\rho \in R_{d+1} \cap F$. Choosing a point $x \in \overline{D}$ such that $\rho = \sigma_x$, we therefore have to describe $T'(\rho)$ as a subcomplex of $\ell k_\Delta(\rho) = \Delta_x \cong \Delta(\Phi_x, \mathbb{F}_q)$ (cf. Corollary 2). This will be done by using the results and the notations of sections 1.3 and 1.4. In the following, the relation "σ is opposite to τ" will be abbreviated by "σ op τ".

LEMMA 8: *Let $\rho \in R_{d+1} \cap F$, $x \in \overline{D}$ and $\overline{\sigma}_y \in \Delta_x$ be given such that $\rho = \sigma_x$ and $\overline{\sigma}_y$ is defined as in Lemma 2. Then one gets*

$$T'(\rho) = \bigcup_{\tau \text{ op } \overline{\sigma}_y} st_{\Delta_x}(\tau) =: \Delta_x^0(\overline{\sigma}_y) .$$

Proof: Set $C := \mathrm{proj}_\rho C_0 \in F$ (this chamber C has to be carefully distinguished from the open cell $C = C_{x,D}$ in section 1.3!) and recall $T'(\rho) = \Gamma_\rho\{\sigma \subsetneq C \setminus \rho\}$ (cf. Lemma 7 (iv)). As a first step in our proof, we have to determine "the position of $|C \setminus \rho|$ relative to $|\rho|$ in $\overline{D}$". Note that $C \setminus \rho$ needn't coincide with the simplex we called κ^- in section 1.3. But in any case, $C \setminus \rho$ contains the simplex $\overline{\sigma}_{y^-}$ of Lemma 3 as the following geometric reasoning shows:

CLAIM: $\mathbf{y^- \in |C|}$. This follows from three simple observations. First of all, $[y^-, x]$ is contained in a closed cell. Hence $\overline{F}_z \supseteq [y^-, x]$ for every $z \in]y^-, x[$. This implies:

(a) $|C'| \cap]y^-, x[\neq \emptyset \implies [y^-, x] \subseteq |C'| \quad \forall$ chambers $C' \in \Sigma_0$.

Next we consider the "convex hull" $\mathcal{C}$ of C_0 and C , i.e. the set of chambers $\mathcal{C} := \mathbf{conv}\{\mathbf{C_0}, \mathbf{C}\} := \{C' \in \Sigma_0 |\ d(C_0, C') + d(C', C) = d(C_0, C\}$. Then $|\mathcal{C}| := \bigcup_{C' \in \mathcal{C}} |C'|$ is an intersection of closed half-spaces (cf. [T1], Theorem 2.19) and hence a convex subset of $|\Sigma_0| = V$. Therefore, we obtain

(b) $[0, x] \subseteq |\mathcal{C}|$.

On the other hand, $d(C_0, C') \geq d(C_0, \rho) = d(C_0, C)$ holds for every chamber C' containing ρ and hence

(c) $x \in |C'| \iff C' = C \quad \forall\, C' \in \mathcal{C}$.

Obviously, (a) – (c) imply $y^- \in |C|$. Therefore, $\sigma_{y^-} \subseteq C$ and $\overline{\sigma}_{y^-} \subseteq C \setminus \rho$. In view of $\rho_x(\Gamma_\rho) = \rho_x(\Gamma_x) = \mathrm{Stab}_{\overline{G}_x}(\overline{\sigma}_y)$ (cf. Corollary 3) and in view of the fact that $\overline{\sigma}_y$ and $\overline{\sigma}_{y^-}$ are opposite in Δ_x (cf. Lemma 3), $\Gamma_\rho\{\sigma \subsetneq C \setminus \rho\} = \bigcup_{\tau \text{ op } \overline{\sigma}_y} st_{\Delta_x}(\tau)$ is now a consequence of the following lemma.

LEMMA 9: *Let A be a group acting type-preservingly and strongly transitively (in the sense of [Br3], section V.1) on a building Θ of spherical type. Then for* *every* $\sigma \in \Theta$, $A_\sigma := \mathrm{Stab}_A(\sigma)$ *acts transitively on the set*

$$\{C \in \Theta |\ C \quad \textit{is a chamber and} \quad \exists\ \tau \subseteq C \quad \textit{such that} \quad \tau \text{ op } \sigma\}$$

Proof: It follows from the assumptions that A_σ acts transitively on the set of all apartments containing σ. Every such apartment contains exactly one simplex opposite to σ. Hence, A_σ acts transitively on the set of all these simplices.

Now let τ be a simplex opposite to σ and let two chambers $C_1, C_2 \in st_\Theta(\tau)$ be given. Then there exists an apartment Σ containing C_1, C_2 and σ: Set $C_2' := \mathrm{proj}_\sigma C_2$. Then $C_2 = \mathrm{proj}_\tau C_2'$ (cf. [T1], Theorem 3.28). Therefore, an apartment Σ containing C_1 and C_2' contains C_2, too. Now we set $N := \mathrm{Stab}_A(\Sigma)$. According to the assumptions, there exists an $n \in N_\tau := \mathrm{Stab}_N(\tau)$ such that $nC_1 = C_2$. But $N_\tau = N_\sigma \subseteq A_\sigma$. ∎

As a first consequence of Lemma 8, we now obtain condition (d) (iii) of Lemma 6:

COROLLARY 5: *There are infinitely many* $d \in \mathbb{N}$ *such that* $|T(\rho)|$ *is non-contractible for at least one* $\rho \in R_{d+1}$.

Proof: If $\dim \rho = n-1$, $\dim \ell k_\Delta(\rho) = 0$. Then Lemma 8 implies $T'(\rho) = \Delta_x \setminus \{\overline{\sigma}_y\}$ which consists of at least two points. Hence $|T(\rho)| = |\partial\rho * T'(\rho)|$ is non-contractible in this case.

Now there are infinitely many d such that R_{d+1} contains elements of dimension $n-1$: Let v be an arbitrary vertex of F, say of type i, such that $st_{\Sigma_0}(v) \subseteq F$. Set $C' := \mathrm{proj}_v C_0$ and let C be the chamber "opposite" to C' in $st_{\Sigma_0}(v)$, i.e. $C \setminus \{v\}$ and $C' \setminus \{v\}$ are opposite in $\ell k_{\Sigma_0}(v)$. Then the usual restriction $R(C)$ of C with respect to C_0 , i.e. the smallest simplex $\sigma \subseteq C$ satisfying $d(\sigma, C_0) = d(C, C_0)$, is obviously the panel of cotype i of C. Finally, $C \in F$ implies $R^\Gamma(C) = R(C)$. ∎

In the following, we fix the notation already introduced in Lemma 8 and use it in order to distinguish certain spherical buildings:

Definition 2: Let Θ be a building of spherical type.

(i) For every simplex $\sigma \in \Theta$, we set $\Theta^0(\sigma) := \bigcup\limits_{\tau \text{ op } \sigma} st_\Theta(\tau)$. Note that $\Theta^0(\emptyset) = \Theta$ is admitted here.

(ii) We say that Θ possesses **property (S)** if $\Theta^0(\sigma)$ is $\dim\Theta$–spherical for every $\sigma \in \Theta$.

Remarks: (i) The well-known Solomon–Tits theorem states that $\Theta = \Theta^0(\emptyset)$ is always spherical. Furthermore, $\Theta^0(\sigma)$ is "highly symmetrical" and "contains almost all of Θ", usually (if Θ is "big enough", "almost all" chambers are opposite to a given chamber). So one may hope that every spherical building possesses property (S) . But this is, unfortunately, not true. For example, if $\Theta = \Delta(A_3, \mathbb{F}_2)$ and $C \in \Theta$ is a chamber, then $|\Theta^0(C)|$ is a torus (cf. [T2], Remarque 16.7.5 or [AA], Example 3.1). Some further counterexamples are listed in [AA] and in [Abr2].

Nevertheless, it seems to be true that Θ always possesses property (S) if Θ is "thick enough", i.e. if every panel is contained in sufficiently many chambers. At least it is possible to verify property (S) for buildings of type $\Theta = \Delta(\Psi, k)$, $\Psi = A_\ell, B_\ell, C_\ell$ or D_ℓ , provided that $\#k$ is big compared with $\ell = \operatorname{rk} \Psi$ (cf. Lemma 10 below).

(ii) I will give a concrete description of $\Theta^0(\sigma)$ in the A_ℓ-case: It is well known that $\Delta(A_\ell, k)$ can be identified with the flag complex $\Theta = \operatorname{Flag} \mathcal{U}$ associated to the poset $\mathcal{U}$ of all non-trivial, proper subspaces of $k^{\ell+1}$. Two vertices U_1, U_2 of Θ are opposite in Θ iff $U_1 \oplus U_2 = k^{\ell+1}$ (consider an apartment containing U_1 and U_2). Therefore, a simplex $\tau = \{T_1, \ldots, T_r\}$ of Θ is opposite to $\sigma = \{S_1, \ldots, S_r\}$ iff $T_i \oplus S_i = k^{\ell+1}$ $(1 \le i \le r)$ for an appropriate numbering of the vertices. This implies

$$\Theta^0(\sigma) = \operatorname{Flag}\{U \in \mathcal{U} \mid (U \cap S_i = 0 \ \vee \ U + S_i = k^{\ell+1}) \ \forall 1 \le i \le r\}.$$

It is shown in [AA] that the complex on the right-hand side is $(\ell - 1)$-spherical if $\#k \ge \sum_{i=1}^{r} \binom{\ell-1}{d_i - 1}$, where $d_i := \dim S_i$.

Hence $\Delta(A_\ell, k)$ possesses property (S) for all fields with at least $2^{\ell-1}$ elements.

(iii) Group theoretically, property (S) admits the following interpretation: Let Θ be the spherical building associated to a root datum $(S, (V_b)_{b \in \Psi})$, Π a base of Ψ , $V^a := \langle V_b \mid b \in (\sum_{a' \in \Pi \setminus \{a\}} \mathbb{N}_0 a') \cap \Psi \rangle$ for $a \in \Pi$, $V := \prod_{b \in \Psi^+} V_b$, $B = SV$ and C the chamber stabilized by B . The following two observations are due to Tits (cf. [T2], section 16):

Assume $\operatorname{rk} \Psi = 2$. Then $|\Theta^0(C)|$ is 0-connected iff $V = \langle V_a \mid a \in \Pi \rangle$.
Assume $\operatorname{rk} \Psi = 3$. Then $|\Theta^0(C)|$ is 1-connected iff V is the amalgamated product of its subgroups V^a, $a \in \Pi$.

More generally, for $\operatorname{rk}\Psi =: m$ $\Theta^0(C)$ is $(m-1)$-spherical iff the system $\{V^a \mid a \in \Pi\}$ is $(m-1)$-generating for V in the sense of [AH], because $\Theta^0(C)$ may be identified with the nerve of the covering

$$V = \bigcup_{\substack{a\in\Pi \\ v\in V}} vV^a.$$

Note that one can replace V, V^a by $B, B^a = SV^a$ in the last statement. Furthermore, similar results may be obtained for $\sigma \subseteq C$, $P = \operatorname{Stab}(\sigma)$ and $\Theta^0(\sigma)$.

(iv) Finally, a trivial remark: In the following, we may concentrate on spherical buildings with connected diagrams, because the join $\Theta_1 * \Theta_2$ possesses property (S) iff Θ_1 and Θ_2 possess property (S). ∎

Now we are interested in spherical buildings of type $\Delta_x \cong \Delta(\Phi_x, \mathbb{F}_q)$ for $x \in V$. Note that the Dynkin diagram $\operatorname{diag}(\Phi_x)$ of Φ_x is a proper subdiagram of the extended Dynkin diagram $\operatorname{diag}(\Phi)^\sim$ (assume $x \in |C_0|$ and consider the base of Φ_x corresponding to the walls of C_0 containing x). Summarizing the results we have put together so far (in particular Lemmata 6 – 8 and Corollary 5), we obtain:

PROPOSITION 1: *Assume that* $\Delta(\Psi, \mathbb{F}_q)$ *possesses property* (S) *for every reduced irreducible root system* Ψ *with* $\operatorname{diag}(\Psi) \subset \operatorname{diag}(\Phi)^\sim$. *Then* $\Gamma = \underline{G}(\mathbb{F}_q[t])$ *is of type* F_{n-1} *but not of type* FP_n.

2.4 CONCLUSIONS. In order to apply Proposition 1, one needs some information about the homotopy type of $|\Theta^0(\sigma)|$ for $\Theta = \Delta(\Psi, \mathbb{F}_q)$ and $\sigma \in \Theta$. As already announced, this problem will be treated in detail elsewhere. Therefore, I will only list the consequences of [Abr2] and [AA] (see also Remark (ii) above), as far as they are relevant to us here. I only mention in passing that results concerning more general buildings of type C_ℓ may also be found in [Abr2] and that, amazingly enough, the D_ℓ-case is much more difficult.

LEMMA 10: $\Delta(\Psi, k)$ *possesses property* (S) *if*

(i) $\Psi = A_\ell$ *and* $\#k \geq 2^{\ell-1}$,

(ii) $\Psi = B_\ell$ *and* $\#k \geq 2^{2\ell-1}$,

(iii) $\Psi = C_\ell$ *and* $\#k \geq 2^{2\ell-2}$,

(iv) $\Psi = D_\ell$ *and* $\#k \geq 2^{2\ell-1}$.

Together with Proposition 1 this implies:

THEOREM 1: *The following groups are of type* F_{n-1} *but not of type* FP_n*:*

(i) $\mathrm{SL}_{n+1}(\mathbb{F}_q[t])$ *provided that* $q \geq 2^{n-1}$,

(ii) $\mathrm{Spin}_{2n+1}(\mathbb{F}_q[t])$ *provided that* $q \geq 2^{2n-1}$,

(iii) $\mathrm{Sp}_{2n}(\mathbb{F}_q[t])$ *provided that* $q \geq 2^{2n-2}$,

(iv) $\mathrm{Spin}_{2n}(\mathbb{F}_q[t])$ *provided that* $q \geq 2^{2n-1}$.

∎

Final Remarks: (i) Theorem 1 (i) is the main result of [Ab2]. Using a concreter model of the Bruhat–Tits building in terms of classes of lattices and a filtration especially adapted to this case, a quantitatively slightly better version with boundary condition $q \geq \max_{k=0}^{n-1} \binom{n-1}{k}$ was proved in [Abr1].

(ii) It is easy to get rid of the simple-connectivity of the algebraic groups in Proposition 1 and in Theorem 1:

Let $\underline{G}'$ be an arbitrary Chevalley group of type Φ . Consider a central isogeny $f : \underline{G} \longrightarrow \underline{G}'$. Because f maps S-arithmetic subgroups of $\underline{G}$ onto S-arithmetic subgroups of $\underline{G}'$ (this is well known, cf. for example [M], Corollary 3.2.9), $f(\Gamma)$ and $\underline{G}'(\mathbb{F}_q[t])$ are commensurable. In particular, $\underline{G}(\mathbb{F}_q[t])$ and $\underline{G}'(\mathbb{F}_q[t])$ possess the same finiteness properties.

(iii) Of course, one can also pass over to non-simple Chevalley groups now, provided that they do not contain any factor of type E_6, E_7, E_8 or F_4 (the case G_2 is settled in [Be2]). This is due to the fact that a direct product $\Gamma = \Gamma_1 \times \cdots \times \Gamma_r$ is of type FP_m iff all factors Γ_i are. ∎

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